Mathematics for Engineers II. lectures

Pál Burai

Power series, Fourier series

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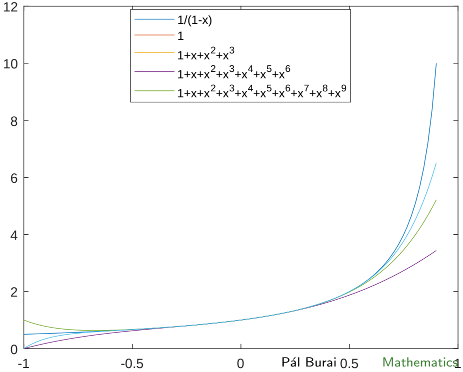
Power series, Taylor series

It is a well-known fact, that

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

if $-1 < x < 1$.
On the left hand side of the equation there is sum containing an infinite number of powers of x. On the right hand side there is a usual expression of a function of the unknown variable of x.



Power series, Taylor series

We are looking for such functions which can be represented in this form, that is to say, as an infinite sum of powers of the unknown variable.
In general, the "infinite polynomial"

$$a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

is said to be a **power series**.
The finite sum of the power series

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = \sum_{i=0}^{n} a_i x^i$$

is called the n**. partial sum**.
If a function f can be written in this form then we say that f **can be represented as a power series**.

Important remark

There are functions which cannot be represented as a power series.

Example

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

This function is infinitely many times differentiable, but it cannot be expressed as a power series.

Application of power series

- **Evaluation.** The functional values of exponential, trigonometric and logarithmic functions, for instance, can be computed numerically with the help of power series to a high degree of accuracy.
- **Approximation.** The first terms of a power series can be used to obtain an approximate value for a given function.
- **Term-by-term integration.** It is not always possible to integrate a function as it stands. If, however, the function can be represented by an absolutely convergent power series it can then be integrated term by term to give the value of the integral to a high degree of accuracy.

Power series, Taylor series

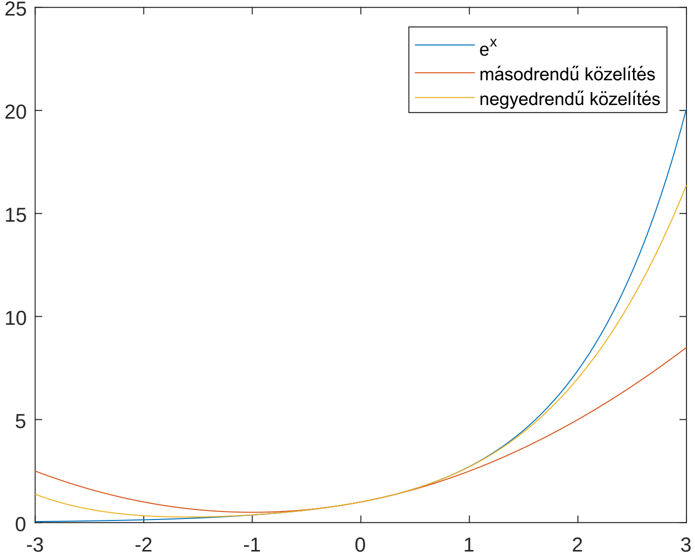
Several functions known from real analysis can be represented as a power series, that is to say, they can be written in the following form:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n.$$

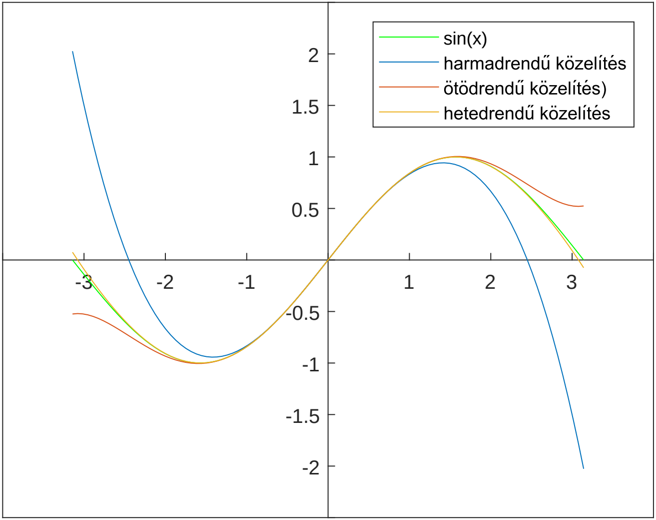
Examples

- $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
- $$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
- $$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Partial sums of the power series of function e^x:



Partial sums of the power series of function $\sin(x)$:



Power series, Taylor series

Absolutely convergent series can be differentiated term-by-term, that is

$$f'(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)' = (a_0 + a_1 x + a_2 x^2 + \cdots)' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$f''(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)'' = (a_0 + a_1 x + a_2 x^2 + \cdots)'' = 2a_2 + 6a_3 x + 12a_4 x^2 \cdots$$

$$f'''(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)' = (a_0 + a_1 x + a_2 x^2 + \cdots)' = 6a_3 + 24a_4 x + 60a_5 x^2 + \cdots$$

With the substitution $x = 0$ we have

$$f(0) = a_0, \quad f'(0) = a_1, \quad \frac{1}{2} f''(0) = a_2, \ldots \frac{1}{n!} f^{(n)}(0) = a_n,$$

so

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

This power series is called the **Maclaurin series of** f.

There are function for which Maclaurin's series converge for values of x within a certain range. This range is referred to as the **interval of convergence** of the power series.

Example

The geometric series

$$1 + x + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$$

is only convergent provided that $-1 < x < 1$.

Definition

The ration

$$R := \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

is called the **radius of convergence** of the series

$$\sum_{n=0}^{\infty} a_n x^n.$$

Power series, Taylor series

Theorem

Let's denote by R the radius of convergence of the series

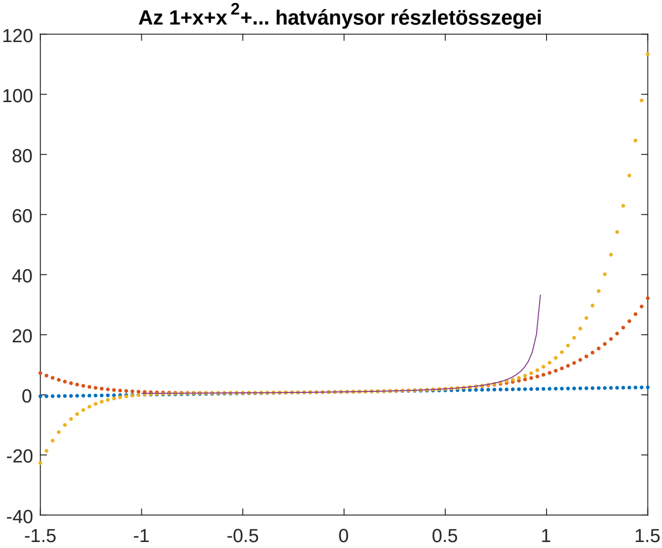
$$\sum_{n=0}^{\infty} a_n x^n.$$

Then the power series is absolutely convergent for all $|x| < R$, it is divergent for all $|x| > R$.

Exercise

Obtain the radius of convergence of the following series!

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad \frac{1}{1-3x} = \sum_{n=0}^{\infty} 3^n x^n.$$


Az 1+x+x 2+... hatványsor részletösszegei
120
100
80
60
40
20
0
-20
-40
-1.5
-1
-0.5
0
0.5
1
1.5

Approximate values of functions with Maclaurin series

Assume that the radius of convergence of a power series is R, and its sum is $f(x)$, if $|x| < R$. Let $|x_0| < R$. Calculate the approximate value of $f(x_0)$!

$$f(x_0) = \underbrace{a_0 + a_1 x_0 + \cdots + a_n x_0^n}_{f(x_0)\ n\text{th order approximation}} + \underbrace{a_{n+1} x_0^{n+1} + \cdots}_{\text{remainder}}$$

Example

Estimate the value of sin(2)!

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \text{ it follows } \sin(2) \approx 2 - \frac{8}{6} + \frac{32}{120} = \frac{14}{15} = 0.9333$$

The Maclaurin series approximates well only in the neighbourhood of zero. This is why it is often useful to expand a function at an arbitrary position x_0.

Definition

Let's assume that f is infinitely many times differentiable around x_0. If

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$
$$+ \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

then f is said to be **representable as a power series around** x_0. A power series like this is called the **Taylor series of f around** x_0.

Example

Find the Taylor expansion around $x_0 = \frac{\pi}{2}$ of cos function.
All the even order terms can be cancelled because $\cos(\frac{\pi}{2}) = 0$, so

$$\cos(x) = \cos(\tfrac{\pi}{2}) - \sin(\tfrac{\pi}{2})(x - \tfrac{\pi}{2}) - \frac{\cos(\frac{\pi}{2})}{2!}(x - \tfrac{\pi}{2})^2 + \frac{\sin(\frac{\pi}{2})}{3!}(x - \tfrac{\pi}{2})^3 + \cdots = -(x - \tfrac{\pi}{2}) + \frac{1}{3!}(x - \tfrac{\pi}{2})^3 + \cdots$$

Exercise

Approximate the value of e using the Taylor expansion of the exponential function! What will be the "good" x_0?

Power series, Taylor series, Example

The atmospheric pressure p is a function of altitude h and is given by

$$p(h) = p = p_0 e^{-\alpha h},$$

where p_0 and α are constants, p_0 being the pressure when $h = 0$. To calculate the pressure difference we have

$$\Delta p = p(h) - p(0) = p - p_0 = p_0(e^{-\alpha h} - 1).$$

Since $e^{-x} = 1 - x + \cdots$, as a first approximation to the pressure difference, it follows that

$$\Delta p \approx p_0(1 - \alpha h - 1) = -p\alpha h.$$

Suppose we want to calculate the altitude h when the pressure p is decreased by 1% of the pressure at $h = 0$, i.e. $\frac{\Delta p}{p_0} = \frac{-1}{100}$, if $\alpha = 0.121 \cdot 10^{-3}$. Then

$$h \approx -\frac{\Delta p}{p_0} = \frac{1}{100} \frac{1}{0.121 \cdot 10^{-3}} = 82.64m.$$

As we have seen, power series representations of functions make it possible to approximate those functions as closely as we want in intervals near a particular point of interest by using partial sums of the series, that is, polynomials.

In many important applications of mathematics, the functions involved are required to be periodic. For example, much of electrical engineering is concerned with the analysis and manipulation of waveforms, which are periodic functions of time. Polynomials are not periodic functions, and for this reason power series are not well suited to representing such functions.

Much more appropriate for the representations of periodic functions overextended intervals are certain infinite series of periodic functions called Fourier series.

Definition

A **periodic function** f is a function such that $f(x) = f(x + L)$, where L is the smallest value for which the relationship is satisfied. Then L is called the **period** of f.

To simplify the mathematics we will start by considering functions whose period is 2π, this implies that $f(x) = f(x + 2\pi)$.

Example

The function $\sin\colon \mathbb{R} \to [0, 1]$ satisfies the equation $\sin(x) = \sin(x + 2k\pi)$ for an arbitrary integer k. So, it is periodic. The smallest positive integer for which the above equality correct is 2 so the period of sin is 2π.

Fourier series

Definition

Let $a_0, a_1, \ldots, \; b_1, b_2, \ldots$ be numerical sequences, then the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

is called a **Fourier series**.

The terms in Fourier series differ in period (or frequency). The nth term has the period $\frac{2\pi}{n}$. Considering a system of coefficients $a_0, a_1, \ldots, b_1, b_2, \ldots$ for which the sequence is convergent. Then there is a function f such that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)).$$

The series then is said to be the **Fourier series of f**.
One can prove that a series like this can be integrated term-by-term.
Question: How can be determined the system of coefficients for a given function?

For the evaluation of the coefficients we will state the results of some definite integrals in the range $-\pi$ to π, where n and m are positive integers.

$$\int_{-\pi}^{\pi} \cos(nx)dx = \int_{-\pi}^{\pi} \sin(nx)dx = 0$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = \int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = \begin{cases} 0, & \text{if } m \neq n, \\ \pi, & \text{if } m = n, \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)dx = 0.$$

Evaluation of a_0

To find a_0 we integrate the Fourier series from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2}\underbrace{\int_{-\pi}^{\pi} a_0 dx}_{=2\pi a_0} + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-\pi}^{\pi} \cos(nx)dx}_{=0} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-\pi}^{\pi} \sin(nx)dx}_{=0}$$

Therefore, we have obtained

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx.$$

Evaluation of a_k

For a given k multiply the Fourier series by $\cos(kx)$ and integrate from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x)\cos(kx)dx = \frac{a_0}{2}\underbrace{\int_{-\pi}^{\pi}\cos(kx)dx}_{=0} + \sum_{\substack{n=1\\ n\neq k}}^{\infty} a_n \underbrace{\int_{-\pi}^{\pi}\cos(nx)\cos(kx)dx}_{=0}$$

$$+a_k\underbrace{\int_{-\pi}^{\pi}\cos(kx)\cos(kx)dx}_{=\pi} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-\pi}^{\pi}\cos(kx)\sin(nx)dx}_{=0}.$$

We thus obtain

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx.$$

Evaluation of b_k

Proceeding in the same way, multiply the Fourier series by $\sin(kx)$ and integrate from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x)\sin(kx)dx = \frac{a_0}{2}\underbrace{\int_{-\pi}^{\pi}\sin(kx)dx}_{=0} + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-\pi}^{\pi}\cos(nx)\sin(kx)dx}_{=0}$$

$$+\sum_{\substack{n=1\\ n\neq k}}^{\infty} b_n \underbrace{\int_{-\pi}^{\pi}\cos(kx)\sin(nx)dx}_{=0} + b_k \underbrace{\int_{-\pi}^{\pi}\cos(kx)\sin(kx)dx}_{=\pi}.$$

Hence

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx.$$

Fourier series, Evaluation of the coefficients

If a function f of period 2π can be represented in a Fourier series then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx, \quad a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)dx,$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)dx.$$

Since f is a periodic function of period 2π we could, if we wished, use the range 0 to 2π instead, or any other interval of length 2π.

Fourier series, Pointwise convergence criterion

We have not yet discussed the conditions that must be satisfied by f for the expansion to be possible. There are, in fact, several sufficient conditions which guarantee that the Fourier expansion is valid, and most functions the applied scientist is likely to meet in practice will be Fourier expandable.

Dirichlet's lemma

A periodic function f which is bounded, which has a finite number of maxima and minima and a finite number of points of discontinuity in the interval $[-L, L]$ has a convergent Fourier series.
Moreover, this series converges towards the value of the function f at all points where it is continuous. At points of discontinuity the value of the Fourier series is equal to the arithmetical mean of the left-hand and right-hand limit of the function.

Fourier series, Odd and even functions

A function is even when $f(x) = f(-x)$. In this case all the coefficients b_n vanish. Since $f(x)\sin(nx)$ is an odd function, its integral from $-\pi$ to π is zero.

Fourier series of an even function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx).$$

A function is even when $f(x) = -f(-x)$. In this case all the coefficients a_n vanish. Since $f(x)\cos(nx)$ is an odd function, its integral from $-\pi$ to π is zero.

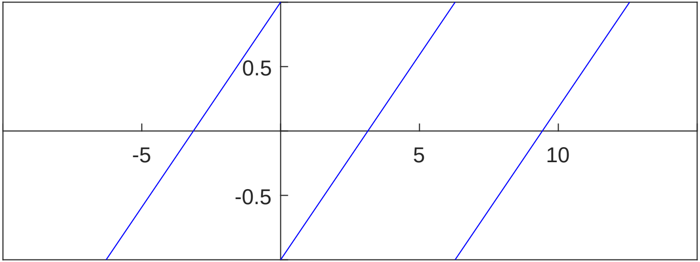
Fourier series of an odd function

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

Sawtooth waveform: The 2π periodic extension of the function which is defined by

$$f(x) = \begin{cases} \frac{1}{\pi}x + 1 & -\pi \le x \le 0, \\ \frac{1}{\pi}x - 1 & 0 < x \le \pi \end{cases}$$

is called the sawtooth waveform function.



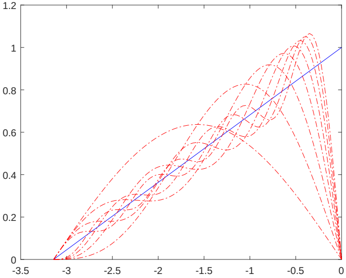
Since the function is odd, only the coefficients b_k are required.

$$b_k = \frac{1}{\pi}\int_{-\pi}^{0}\left(\frac{1}{\pi}x+1\right)\sin(kx)dx + \frac{1}{\pi}\int_{0}^{\pi}\left(\frac{1}{\pi}x-1\right)\sin(kx)dx = -\frac{2}{k\pi}.$$

Hence, for the sawtooth waveform the Fourier series is

$$\frac{-2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}.$$

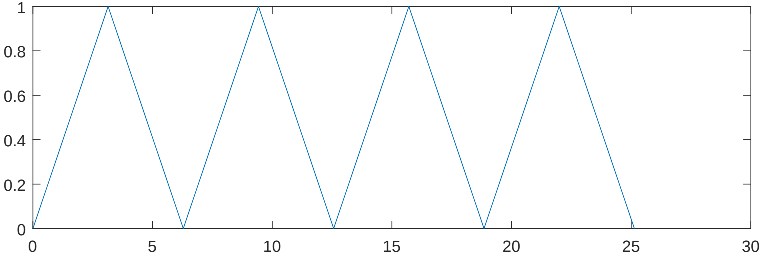
The figure above shows the first eight terms of the expansion in the range $[-\pi, 0]$.



Triangular waveform: The 2π periodic extension of the function which is defined by

$$f(x) = \begin{cases} -x, & -\pi < x \leq 0 \\ x, & 0 < x \leq \pi \end{cases}$$

is said to be the triangular waveform function.



Fourier series, Examples

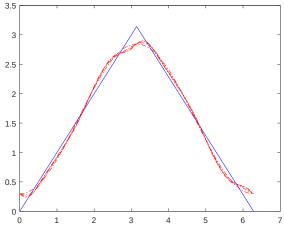
Since triangular waveform is an even function, we only need to calculate the coefficients a_k.

$$a_0 = \pi, \qquad a_k = \begin{cases} 0 & \text{if } k \text{ even,} \\ -\frac{4}{\pi k^2} & \text{if } k \text{ odd.} \end{cases}$$

The Fourier series of the triangular waveform is

$$\frac{\pi}{2} + \frac{-4}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n+1)x)}{(2n+1)^2}.$$

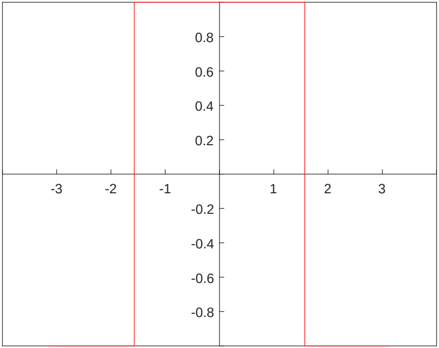
The figure below shows the first four terms of the expansion over the interval $[0, \pi]$



Rectangular waveform: The 2π periodic extension of the function which is defined by

$$f(x) = \begin{cases} -1, & -\pi < x \leq -\frac{\pi}{2} \\ 1, & -\frac{\pi}{2} < x \leq \frac{\pi}{2} \\ -1, & \frac{\pi}{2} < x \leq \pi \end{cases}$$

is the rectangular waveform function.



Fourier series, Examples

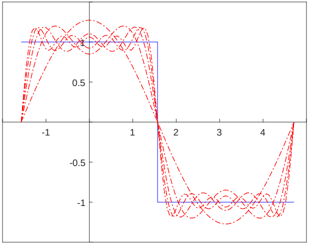
Since the rectangular waveform function is even, we need only calculate the coefficients a_k.

$$a_0 = 0, \qquad a_k = \frac{2}{k\pi} \sin \frac{k\pi}{2}.$$

The Fourier series for the rectangular waveform is

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\sin(n\pi)}{2} \cos(nx).$$

The figure below shows the first four terms of the expansion over the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.



Fourier series, Expansion of functions of period $2L$

Let f be a function of period $2L$, i.e.

$$f(x+2L)=f(x).$$

If we put $z=\frac{\pi}{L}x$, then the new function $\tilde{f}(z)=f\left(\frac{\pi}{L}x\right))$ is a periodic function of period 2π. (For the sake of simplicity we write f instead of $\tilde{f}$). We have

$$f(z)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos(nz)+b_n\sin(nz)).$$

To get back to the original function, we simply replace z by $\frac{\pi}{L}$ and obtain

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi}{L}x\right)+b_n\sin\left(\frac{n\pi}{L}x\right)\right).$$

The coefficients of a Fourier series of a function with period $2L$ are

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx,$$

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{k\pi}{L}x\right)dx,$$

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{k\pi}{L}x\right)dx.$$

Example

Rectangular waveform of period 4

$$f(x) = \begin{cases} -1, & -2 < x \leq 0 \\ 1, & 0 < x \leq 2 \end{cases}$$

Fourier spectrum

A periodic function of time, such as a vibratory motion, expressed as a Fourier series is often represented by a Fourier spectrum. It consists of the values of the amplitudes and phases of the different terms as a function of the frequency.
For example, if s is a vibration or a signal in an electrical system then its Fourier series can take on the following form:

$$s = \sum_{n=1}^{\infty} a_n \sin(n\omega t + \phi_n).$$

Fourier spectrum

$$s = \sum_{n=1}^{\infty} a_n \sin(n\omega t + \phi_n).$$

Then ω is called the **fundamental frequency** and $2\omega,\ 3\omega, \ldots$ are said to be the **harmonics**. The values $a_1,\ a_2, \ldots$ are the **amplitudes** belonging to the fundamental frequency and to the harmonics respectively.
A periodic electrical signal introduced into a transducer, filter or amplifier will be modified; it will be damped and distorted. By expressing the signal as a Fourier series it is easy to find out how each Fourier component is affected.
The Fourier series above is well characterized by the pairs

$$(a_1, \omega), \quad (a_2, 2\omega), \ldots (a_n, n\omega), \ldots$$

which is called the **Fourier spectrum**.
If the spectrum of the input and the output are the same, it means that the signal is faithfully reproduced.

Obtain the Fourier series of the following functions!

1.
$$f(x) = \begin{cases} 0, & -\pi \le x < -\frac{\pi}{2} \\ 1, & -\frac{\pi}{2} \le x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \le x \le \pi \end{cases}$$

2.
$$f(x) = \begin{cases} 1 & -\pi \le x < 0 \\ -1 & 0 \le x \le \pi \end{cases}$$

3.
$$f(x) = |\sin(x)|$$

4.
$$f(x) = \begin{cases} 0 & -2\pi \le x < -\pi \\ 1 & -\pi \le x < \pi \\ 0 & \pi \le x < 2\pi \end{cases}$$